International Journal of Theoretical Physics, Vol. 32, No. 1, 1993

Similarity Solutions of the Cubic Nonlinear Klein–Gordon Equation

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Received April 20, 1992

Using the direct method introduced by Clarkson and Kruskal recently, we obtain the similarity reductions of the cubic nonlinear Klein–Gordon equation when $z=0$.

The standard method for finding similarity reductions of a given partial differential equation (PDE) is to use the method (Olver, 1986) [and/or the nonclassical method (Bluman and Cole, 1969)] of group-invariant solutions, which often involves a large amount of tedious algebra and auxiliary calculations. Clarkson and Kruskal (1981) (CK) presented a simple but powerful direct method for the Boussinesq equation (BE) and other (1+1)-dimensional PDEs. Recently, Lou (1990*a*) further extended the CK method to a (2+1)-dimensional Kadomtsev–Petviashvili equation (KPE). In this note, we apply the CK method to the cubic nonlinear Klein–Gordon equation (KGE)

$$u_{tt}-u_{xx}=\lambda u+\mu u^3 \tag{1}$$

Equation (1) becomes the well-known model when $\lambda<0$, $\mu>0$. Here we wish to discuss the general case that is not confined to $\lambda<0$, $\mu>0$.

All the similarity solutions of the form

$$u(x, t)=U(x, t, w(z(x, t))) \tag{2}$$

where U and z are functions of the indicated variables and $w(z)$ satisfies an ordinary differential equation (ODE), may be obtained by substituting (2) into (1). However, as CK did for the BE and as Lou did for the KPE, we

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0020-7748/93/0100-0039$07.00/0

can prove that it is also sufficient to seek a similarity reduction of the cubic nonlinear KGE in the special form

$$u(x, t) = \alpha(x, t) + \beta(x, t)w(z(x, t)) \tag{3}$$

rather than the most general form (2). When $z_x = 0$, we still can select a similarity reduction of the cubic nonlinear KGE in the special form

$$u(x, t) = \alpha(x, t) + \beta(x, t)w(z(t)) \tag{4}$$

rather than the most general form

$$u(x, t) = U(x, t, w(z(x, t))) \tag{5}$$

It is clear that $z(t)$ may be taken as t simply, without loss of generality.

Substituting equation (4) with $z = t$ into equation (1) yields

$$\begin{aligned} &\beta w'' + 2\beta_t w' + (\beta_{tt} - \beta_{xx} - \lambda\beta - 3\mu\alpha^2\beta)w - 3\mu\alpha\beta^2 w^2 - \mu\beta^2 w^3 \\ &\quad + (\alpha_{tt} - \beta_{xx} - \lambda\alpha - \mu\alpha^3) = 0 \end{aligned} \tag{6}$$

We use the coefficient of w'' as the normalizing coefficient. For this to be an ODE, we have

$$2\beta_t = \beta\Gamma_1(t) \tag{7}$$

$$\beta_{tt} - \beta_{xx} - \lambda\beta - 3\mu\alpha^2 = \beta\Gamma_2(t) \tag{8}$$

$$-\mu\beta^3 = \beta\Gamma_3(t) \tag{9}$$

$$-3\mu\alpha^2\beta = \beta\Gamma_4(t) \tag{10}$$

$$\alpha_{tt} - \alpha_{xx} - \lambda\alpha - \mu\alpha^3 = \beta\Gamma_5(t) \tag{11}$$

where Γ_1, Γ_2, Γ_3, Γ_4, and Γ_5 are some function of t. From equation (7) we have

$$\beta = \beta_1(x) \exp \int_0^t \tfrac{1}{2}\Gamma_1(t_1)\, dt_1 \equiv \beta_1(x)\Gamma_{1a}(t) \tag{12}$$

Γ_{1a} may be taken by the replacement

$$w(t) \to w(t)/\Gamma_{1a}(t)$$

and then we have

$$\beta = \beta_1(x), \qquad \Gamma_1(t) = 0, \qquad \Gamma_{1a}(t) = 1 \tag{13}$$

There are different cases to discuss.

Suppose $\beta_1 = \text{const} = 1$. In this case, substituting equation (13) into equations (8)–(11), we get

$$\alpha = \alpha_1(t) \tag{14}$$

$$\Gamma_3 = -\mu, \qquad \Gamma_4 = -3\mu\alpha_1, \qquad \Gamma_2 = -\lambda - 3\mu\alpha_1^2, \qquad \Gamma_5 = 0 \tag{15}$$

where $\alpha_1(t)$ is determined by

$$d^2\alpha_1/dt^2 - \lambda\alpha_1 - \mu\alpha_1^3 = 0 \tag{16}$$

From equation (16), we have

$$(d\alpha_1/dt)^2 = a + \lambda\alpha_1^2 + \tfrac{1}{2}\mu\alpha_1^4 \tag{17}$$

where a is an integration constant.

If we write equation (17) as

$$(d\alpha_1/dt)^2 = (A + B\alpha_1^2)^2 \tag{18}$$

then we obtain

$$\alpha_1(t) = \frac{A}{B} tg(ABt + \theta_0) \tag{19}$$

where θ_0 is another integration constant, and $A = \frac{1}{2}\mu/\lambda$ and $B = \lambda$.

Finally, we have

$$\alpha(x, t) = \alpha_1 + w(t) \tag{20}$$

where $w(t)$ satisfies the following equation:

$$w'' - (\lambda + 3\mu\alpha_1^2)w - 3\mu\alpha_1 w^2 - \mu w^3 = 0 \tag{21}$$

Now we discuss a special case. When $\alpha_1 = 0$, equation (21) becomes

$$w'' - \lambda w - \mu w^3 = 0 \tag{22}$$

for which one can obtain a general solution according to the values of λ and μ.

In summary we have reduced the cubic nonlinear KGE to a type of OPE. A question remains of how to get the general solution of equation (21) and obtain the similarity reductions of the cubic nonlinear KGE just as equation (3) gives the similarity reductions of the high-order KdV. We leave this to further work.

REFERENCES

Bluman, G. W., and Cole, J. D. (1969). *Journal of Mathematics and Mechanics*, **18**, 1025.
Clarkson, P. A., and Kruskal, M. D. (1989). *Journal of Mathematical Physics*, **30**, 2201.
Lou, S. Y. (1990*a*). *Physics Letters A*, **151**, 133.
Lou, S. Y. (1990*b*). *Journal of Physics A: Mathematical and General*, **23**, L649.
Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations*, Springer, Berlin.